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HANDLEBODIES AND p-CONVEXITY

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The aim of this paper is to study the Riemannian geometry of manifolds with boundary. In a previous paper [4], the author proved the following theorem.

Let M be a compact connected manifold with nonempty boundary. If M admits a Riemannian metric with nonnegative sectional curvature and p-convex boundary, then M has the homotopy type of a CW-complex of dimension $\leqslant p-1$.

Note. The author has recently learned that this theorem has also been proved independently by H. Wu [5].

One of the main results of this paper is a converse of this theorem.

We begin by recalling the notion of p-convexity. Let X be an $(n-1)$-dimensional (normally oriented) hypersurface in a Riemannian manifold Ω and let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n-1}$ be its principal curvature functions. X is called p-convex if $\lambda_1 + \cdots + \lambda_p > 0$ at each point of X. Note in particular that "1-convexity" is the usual notion of convexity; "$(n-1)$-convexity" means that X has positive mean curvature. Also note that p-convexity implies $(p+1)$-convexity.

In [3], by a handle-attaching process, Lawson and Michelsohn showed the following: *Suppose X has positive mean curvature and let X' be a hypersurface obtained from X by attaching an ambient k-handle to the positive side of X. If the codimension $(n-k)$ of the handle is $\geqslant 2$, then X' can be constructed also to have positive mean curvature.* (That is to say that X' is ambiently isotopic to a hypersurface of positive mean curvature.)

Our central result is a generalization of this theorem to the p-convex case. Specifically we shall prove the following.

Theorem 1. *Let X be a (normally oriented) p-convex hypersurface in a Riemannian manifold Ω, and let X' be a hypersurface obtained from X by attaching a k-handle D^k to the positive side of X. If $k \leqslant p-1$, then X' can be constructed also to be p-convex.*

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Arguing as in [3] we get the following.

Corollary 2. *Let X be a compact manifold embedded as the boundary of a domain D in a Riemannian manifold Ω. Orient X with respect to the inward pointing normal vector. If D is diffeomorphic to a handlebody of dimension $\leqslant p-1$, then X is ambiently isotopic through mutually disjoint embeddings to a p-convex hypersurface X' in Ω. The new hypersurface X' bounds a domain D' which is diffeomorphic to D.*

Applying this together with the fundamental results of Gromov in [1] we then obtain the following result which is a converse to the theorem in [4].

Theorem 3. *Let M be a compact connected manifold with nonempty boundary. If M is a handlebody with handles only of dimension $\leqslant p-1$, then M supports a Riemannian metric with positive sectional curvature and p-convex boundary.*

In fact, by the theorem of Gromov the sectional curvature of M can be ε-pinched for any $\varepsilon > 0$. If M is parallelizable, then by immersion-submersion theory (cf. [2]) there exists an immersion $M \hookrightarrow S^n(1)$ where $n = \dim M$. By pulling back the constant curvature metric from $S^n(1)$ and proceeding as in Theorem 3, we have the following.

Theorem 4. *Let M be as in Theorem 3. If M is parallelizable and is a handlebody with handles only of dimension $\leqslant p-1$, then M supports a Riemannian metric with constant sectional curvature 1 and p-convex boundary.*

The remainder of the paper is devoted to proving Theorem 1. Since our basic set-up closely follows Lawson and Michelsohn [3], our presentation will be brief. The basic picture is shown in Figure 1.

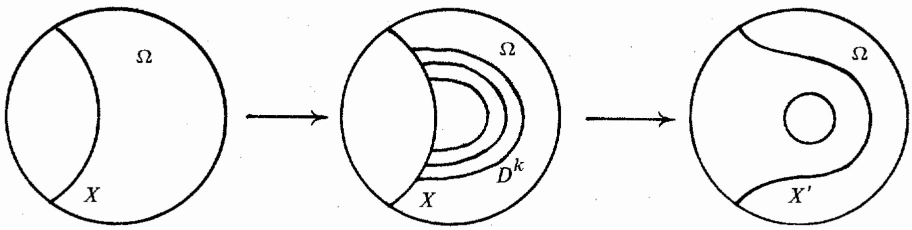

FIGURE 1

1. The basic set-up

Assume Ω is connected. Let X be as in Theorem 1. Positive mean curvature (implied by p-convexity) implies a well-defined normal direction to X; i.e., we have an embedding of $X \times (-1,1)$ in Ω with the image of $X \times 0$ identified to X. Let X^+ be the union of components of $\Omega \setminus X$ containing $X \times (0,1)$, and X^- be the union of components of $\Omega \setminus X$ containing $X \times (-1,0)$.

Let D^k be a k-dimensional disk orthogonally attached to X in X^+. Set, for $x \in \Omega$,

$$s(x) \equiv \operatorname{dist}(x, X), \qquad r(x) \equiv \operatorname{dist}(x, D^k).$$

Then there exists a neighborhood Ω_1 of X in Ω such that s is smooth in $\Omega_1' \equiv \Omega_1 \setminus X^-$ and $\|\nabla s\| \equiv 1$. Similarly, there exists a neighborhood Ω_2 of D^k such that r is smooth in $\Omega_2' \equiv \Omega_2 \setminus (X^- \cup D^k)$ and $\|\nabla r\| \equiv 1$. Then $r^{-1}(r_0) \cap \Omega_2'$ is a hypersurface in Ω_2' for any sufficiently small $r_0 > 0$.

Hence, the map

$$(r, s)\colon \Omega_1' \cap \Omega_2' \to \mathbf{R}^2$$

is a smooth submersion. Our idea is to construct a regular curve γ which is essentially the graph of some function $s = f(r)$ in $\mathbf{R}^2$, so that the hypersurface $S_\gamma \equiv (r, s)^{-1}(\gamma)$ joins $r^{-1}(\varepsilon_0)$ to X smoothly for some $\varepsilon_0 > 0$, and the whole new hypersurface obtained will still be p-convex.

Recall that the second fundamental form of the level hypersurface of a function is closely related to its Hessian form. We summarize this fact in the following.

Lemma 1. *Let u be a smooth function on a domain of Ω. Then at every point the 2-form $\nabla^2 u$ defined by*

$$\nabla^2 u(\cdot, \cdot) = \operatorname{Hess}_u(\cdot, \cdot) = \langle \nabla \cdot (\nabla u), \cdot \rangle$$

is symmetric. Furthermore, if $\|\nabla u\| \equiv 1$, then ∇u lies in the null space of $\nabla^2 u$, and when restricted to $\nabla u^\perp$, $\nabla^2 u$ is the second fundamental form of the level hypersurface of u with respect to $-\nabla u$.

Proof. See [3]. q.e.d.

Suppose u is a function as in Lemma 1. Let

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$$

be the eigenvalues of $\nabla^2 u$. We denote by $\sigma_u(m)$ the sum $\lambda_1 + \cdots + \lambda_m$ for $m = 1, \cdots, n$.

Remark. Note that by Lemma 1, ∇u is an eigenvector of $\nabla^2 u$, the corresponding eigenvalue is 0. The other $(n-1)$ eigenvalues are the principal curvatures of the level hypersurface of u. We then clearly have that the level hypersurface is p-convex if and only if $\sigma_u(p+1)$ is positive.

Lemma 2. (i) *We can choose Ω_1 such that there exists a constant $\delta > 0$ for which $\sigma_s(p+1) > \delta$ in Ω_1. (Here δ could be replaced by a smooth positive function.)*

(ii) *We can choose Ω_2 such that $\sigma_r(p+1) > c/r$ in $\Omega_2 \setminus (X^- \cup D^k)$, where $c > 0$ is a constant.*

Proof. (i) is from the p-convexity of X.

(ii) is by a calculation in Fermi coordinates and the fact that $k \leqslant p - 1$ as follows.

Choose locally smooth orthonormal vector fields $e_1,\cdots,e_n$ along D^k such that $e_1,\cdots,e_k$ are tangent to D^k and that $e_{k+1},\cdots,e_n$ are normal to D^k. Then for $\xi \in D^k$, $(x_1,\cdots,x_{n-k}) \in \mathbf{R}^{n-k}$ with $x_1^2 + \cdots + x_{n-k}^2$ small, the map

$$(\xi,(x_1,\cdots,x_{n-k})) \mapsto \exp_\xi(x_1 e_{k+1} + \cdots + x_{n-k}e_n)$$

gives a local coordinate in some open set $W \subset \Omega_2$. Extend $e_1,\cdots,e_n$ to smooth vector fields $\tilde{e}_1,\cdots,\tilde{e}_n$ on W, where each $\tilde{e}_i$ is obtained by parallel translation of e_i along the geodesic

$$\alpha(t) = \exp_\xi[t(x_1 e_{k+1} + \cdots + x_{n-k}e_n)], \qquad 0 \leqslant t \leqslant 1.$$

On W, it is clear that

$$r(\xi,(x_1,\cdots,x_{n-k})) = \sqrt{x_1^2 + \cdots + x_{n-k}^2}$$

and that

$$\nabla r = \frac{1}{r}(x_1\tilde{e}_{k+1} + \cdots + x_{n-k}\tilde{e}_n).$$

If the metric were Euclidean, i.e., if all the $\tilde{e}_i$'s were parallel, we would obviously have

$$\sigma_r(p+1) = (p-k)/r.$$

In general, let $V_1,\cdots,V_{p+1}$ be arbitrary $(p+1)$ orthonormal tangent vectors at some point in W. We have that

$$\sum_{i=1}^{p+1} \nabla^2 r(V_i,V_i) = \sum_{i=1}^{p+1} \tilde{\nabla}^2 r(V_i,V_i)$$

$$(*) \qquad + \sum_{i=1}^{p+1}\left(\frac{x_1}{r}\langle \nabla_{V_i}\tilde{e}_{k+1}, V_i\rangle + \cdots + \frac{x_{n-k}}{r}\langle \nabla_{V_i}\tilde{e}_n, V_i\rangle\right),$$

where $\tilde{\nabla}^2 r$ denotes the Hessian of r under the Euclidean metric. Then the first sum in $(*)$ is $\geqslant (p-k)/r$. The second sum in $(*)$ can clearly be bounded by some fixed constant which is independent of r. Therefore by choosing Ω_2 properly and noting that $p - k \geqslant 1$, there exists a constant $c > 0$ such that

$$\sigma_r(p+1) > c/r$$

in $\Omega_2 \setminus (X^- \cup D^k)$.

2. The bending function

Let δ, ε_1, ε_2, and c_0 be fixed positive constants. Our aim in this section is to construct a smooth function f which is defined on $r > \varepsilon_0$ for some $0 < \varepsilon_0 < \varepsilon_1$ such that

$$f(r) = 0 \qquad \text{for } r \geqslant \varepsilon_1;$$

$$f'(r) \leqslant 0 \qquad \text{for } r > \varepsilon_0;$$

$$f(r) \to \varepsilon_3 < \varepsilon_2 \qquad \text{as } r \to \varepsilon_0^+.$$

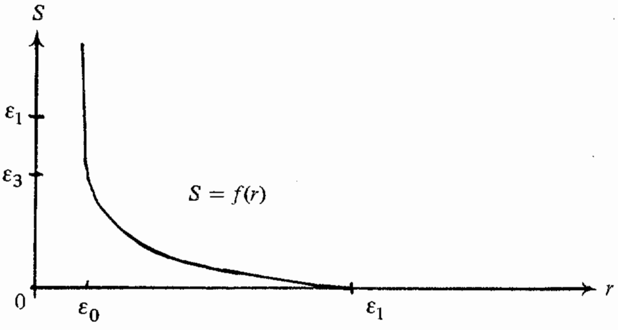

FIGURE 2

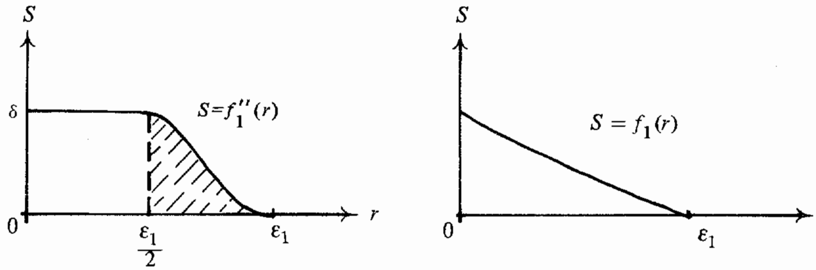

FIGURE 3

All the derivatives of $f \to \infty$ in absolute value as $r \to \varepsilon_0^+$ (see Figure 2). Furthermore, f satisfies either of the following conditions for $r > \varepsilon_0$:

$$\delta - f''(r) - \frac{c_0 f'(r)}{r} > 0 \quad \text{or} \quad \delta - \frac{f''(r)}{f'(r)^2} - \frac{c_0 f'(r)}{r} > 0.$$

We begin by choosing f_1'' properly to get a smooth function f_1 such that

$$\begin{aligned} &f_1(r) = 0 && \text{for } r \geqslant \varepsilon_1; \\ &f_1'(r) \leqslant 0 && \text{for all } r; \\ &0 < f_1''(r) = \text{constant} < \delta && \text{for } r < \varepsilon_1/2; \end{aligned}$$

$$\exp\left[\frac{1}{2c_0 f_1'(\varepsilon_1/2)^2}\right] > 1; \qquad \frac{c_0\left[-f_1'(\varepsilon_1/2)\right]^3}{f_1''(0)} < \frac{\varepsilon_1}{2};$$

$$f_1(0) + \frac{c_0\left[-f_1'(\varepsilon_1/2)\right]^3}{f_1''(0)} \cdot \frac{1}{l}\int_1^l \frac{dt}{\sqrt{2c_0 \ln t}} < \varepsilon_2 \quad \text{for all } l > 1.$$

All the requirements can be satisfied by choosing $f_1''(0)$ small and then by choosing the area of the shaded part in Figure 3 small and also by noting that

$$\frac{1}{l}\int_1^l \frac{dt}{\sqrt{2c_0 \ln t}} \to 0 \quad \text{as } l \to \infty;$$

therefore, in particular, it is bounded for $l > 1$.

Now set

$$a = \exp\left[\frac{1}{2c_0 f_1'(\varepsilon_1/2)^2}\right], \qquad \varepsilon_0 = \frac{c_0\left[-f_1'(\varepsilon_1/2)\right]^3}{a f_1''(0)}.$$

Then $a > 1$ and $a\varepsilon_0 < \varepsilon_1/2$ by the construction of f_1.

Define for $r > \varepsilon_0$

$$f_2(r) = \int_r^{a\varepsilon_0} \frac{dt}{\sqrt{2c_0 \ln(t/\varepsilon_0)}}.$$

We have

$$f_2'(r) = -\frac{1}{\sqrt{2c_0 \ln(r/\varepsilon_0)}}, \qquad f_2''(r) = \frac{1}{2\sqrt{2c_0}\left(\ln(r/\varepsilon_0)\right)^{3/2} r}.$$

Hence

$$\frac{f_2''(r)}{f_2'(r)^2} + \frac{c_0 f_2'(r)}{r} = 0.$$

Finally, let

$$f_3(r) = \begin{cases} f_1(\varepsilon_1/2) + f_2(r) & \text{for } \varepsilon_0 < r \leqslant a\varepsilon_0, \\ f_1(r - a\varepsilon_0 + \varepsilon_1/2) & \text{for } r \geqslant a\varepsilon_0. \end{cases}$$

Then it is easy to verify that f_3 is C^2 and satisfies all the conditions required for f. In fact when $r \geqslant a\varepsilon_0$

$$\delta - f_3''(r) - \frac{c_0 f_3'(r)}{r} > 0$$

by the construction of f_1 and when $\varepsilon_0 < r \leqslant a\varepsilon_0$

$$\delta - \frac{f_3''(r)}{f_3'(r)^2} - \frac{c_0 f_3'(r)}{r} = \delta - \frac{f_2''(r)}{f_2''(r)^2} - \frac{c_0 f_2'(r)}{r} = \delta > 0.$$

The required f is then gotten by a smoothing of f_3.

3. The construction of X'

Let $D_\varepsilon = \{x \in \Omega\colon\ r(x) < \varepsilon\}$ and $X_\varepsilon = \{x \in \Omega\colon\ s(x) < \varepsilon\}$ be tubular neighborhoods of D^k and X respectively.

There exist $\varepsilon_1, \varepsilon_2 > 0$ such that $D_{2\varepsilon_1} \subset \Omega_2$, $X_{2\varepsilon_2} \subset \Omega_1$ and such that $|\langle \nabla r, \nabla s\rangle| < 1$ in $U = \{x \in D_{2\varepsilon_1} \cap X_{2\varepsilon_2} \cap X^+\colon\ r(x) > 0\}$.

Let γ be the curve $s = f(r)$ as in Figure 2. The hypersurface $S_\gamma = (r, s)^{-1}(\gamma)$ smoothly joining $X \setminus (X \cap U)$ to $\partial D_{\varepsilon_0} \setminus (\partial D_{\varepsilon_0} \cap U)$ produces a new hypersurface which will be our hypersurface X' obtained from X by attaching the handle D^k (see Figure 4).

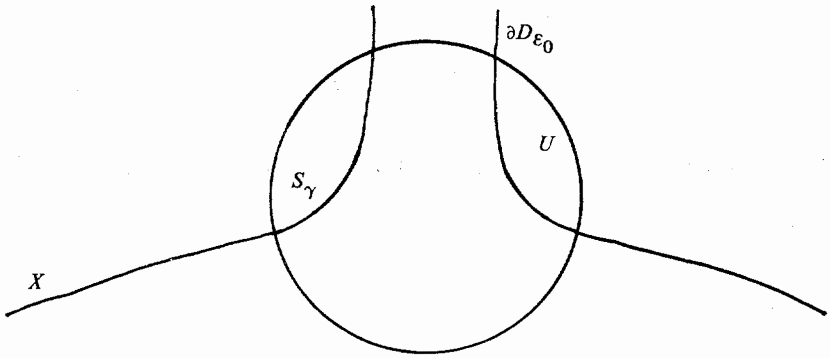

FIGURE 4

We claim that X' is p-convex. It only needs to be verified at the part of S_γ where $r > \varepsilon_0$. For this part, S_γ is the level set of the smooth function $F(x) = s(x) - f(r(x))$.

We have

$$\nabla F = \nabla s - f'(r)\nabla r,$$
$$\nabla^2 F = \nabla^2 s - f'(r)\nabla^2 r - f''(r)(\nabla r)^2.$$

Let $e_n = \nabla F/\|\nabla F\|$. The second fundamental form of S_γ is given by

$$B_F(\cdot,\cdot) = \langle \nabla \cdot e_n, \cdot \rangle = \frac{\nabla^2 F}{\|\nabla F\|} - \frac{\nabla\left(\|\nabla F\|^2\right)\nabla F}{2\|\nabla F\|^3}.$$

Clearly $B_F(e_n, e_n) = 0$ and

$$\begin{aligned}
\nabla_{e_n}\left(\|\nabla F\|^2\right)\nabla_{e_n} F &= \nabla_{e_n}\left(\|\nabla F\|^2\right)\langle e_n, \nabla F\rangle \\
&= \frac{\nabla_{\nabla F}}{\|\nabla F\|}\langle \nabla F, \nabla F\rangle \cdot \left\langle \frac{\nabla F}{\|\nabla F\|}, \nabla F \right\rangle \\
&= 2\langle \nabla_{\nabla F}\nabla F, \nabla F\rangle = 2\nabla^2 F(\nabla F, \nabla F) \\
&= 2\left[\nabla^2 s(\nabla F, \nabla F) - f'(r)\nabla^2 r(\nabla F, \nabla F)\right] - 2\|\nabla F\|^2 f''(r)\left(\nabla_{e_n} r\right)^2 \\
&= 2\left[f'(r)^2\nabla^2 s(\nabla r, \nabla r) - f'(r)\nabla^2 r(\nabla s, \nabla s)\right] - 2\|\nabla F\|^2 f''(r)\left(\nabla_{e_n} r\right)^2,
\end{aligned}$$

where the last equality is obtained by recalling that ∇s is in the null space of $\nabla^2 s$ and that ∇r is in the null space of $\nabla^2 r$.

Then

$$\frac{\nabla_{e_n}(\|\nabla F\|^2)\nabla_{e_n}F}{2\|\nabla F\|^3} = -\frac{f''(r)}{\|\nabla F\|}(\nabla_{e_n}r)^2 - \frac{1}{\|\nabla F\|^3}\left[f'(r)\nabla^2 r(\nabla s,\nabla s) - f'(r)^2\nabla^2 s(\nabla r,\nabla r)\right].$$

Now suppose that $e_1,\cdots,e_p$ are orthonormal vectors tangent to s_γ. Then $\nabla_{e_i}F = 0$ for $i = 1,\cdots,p$.

Therefore

$$\begin{aligned}
\sum_{i=1}^{p} B_F(e_i,e_i) &= \sum_{i=1}^{p} B_F(e_i,e_i) + B_F(e_n,e_n)\\
&= \frac{1}{\|\nabla F\|}\sum_{i=1}^{p}\left[\nabla^2 s(e_i,e_i) - f'(r)\nabla^2 r(e_i,e_i) - f''(r)(\nabla_{e_i}r)^2\right]\\
&\quad + \frac{1}{\|\nabla F\|}\left[\nabla^2 s(e_n,e_n) - f'(r)\nabla^2 r(e_n,e_n) - f''(r)(\nabla_{e_n}r)^2\right]\\
&\quad + \frac{1}{\|\nabla F\|} f''(r)(\nabla_{e_n}r)^2\\
&\quad + \frac{1}{\|\nabla F\|^3}\left[f'(r)\nabla^2 r(\nabla s,\nabla s) - f'(r)^2\nabla^2 s(\nabla r,\nabla r)\right]\\
&\geqslant \frac{1}{\|\nabla F\|}\left[\sigma_s(p+1) - f'(r)\sigma_r(p+1) - f''(r)\sum_{i=1}^{p}(\nabla_{e_i}r)^2\right]\\
&\quad + \frac{1}{\|\nabla F\|^3}\left[f'(r)\nabla^2 r(\nabla s,\nabla s) - f'(r)^2\nabla^2 s(\nabla r,\nabla r)\right]\\
&\geqslant \frac{1}{\|\nabla F\|}\left[\delta - f'(r)\left(\frac{c}{r} - \frac{1}{\|\nabla F\|^2}\left|\frac{r\nabla^2 r(\nabla s,\nabla s)}{r}\right|\right.\right.\\
&\quad \left.\left. - \frac{1}{\|\nabla F\|^2}\left|f'(r)\nabla^2 s(\nabla r,\nabla r)\right|\right) - f''(r)\sum_{i=1}^{p}(\nabla_{e_i}r)^2\right].
\end{aligned}$$

Note that

$$\lim_{r\to 0} r\nabla^2 r(\nabla s,\nabla s) = 0$$

in U, and that

$$\nabla^2 s(\nabla r, \nabla r), \quad \frac{f'(r)}{\|\nabla F\|^2} = \frac{f'(r)}{1 + f'(r)^2 - 2f'(r)\langle \nabla r, \nabla s \rangle}$$

are bounded in U. It is then easy to see that we can choose ε_1, ε_2, c_0 so that

$$\sum_{i=1}^{p} B_F(e_i, e_i) \geqslant \frac{1}{\|\nabla F\|}\left[\delta - \frac{c_0 f'(r)}{r} - f''(r)\right]$$

or (note that $\nabla_{e_i} r = \nabla_{e_i} s / f'(r)$)

$$\sum_{i=1}^{p} B_F(e_i, e_i) \geqslant \frac{1}{\|\nabla F\|}\left[\delta - \frac{c_0 f'(r)}{r} - \frac{f''(r)}{f'(r)^2}\right].$$

Therefore by the construction of f, s_γ is p-convex.

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